Rank-Stratified Projections of a Balanced Orthogonal Cartan Cell and a Codimension-Two Matrix-Pencil Obstruction

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Abstract

Let $J = \mathrm{diag}(I_6, -I_6)$ and write the positive symmetric part of $O(6,6)$ in graph coordinates using strict contractions $Z \in M_6(\mathbb{R})$. A fixed axis–channel block projection defines a map $\Phi(Z) = (Q_H, Q_W)$ into two symmetric 4×4 matrices. We determine the maximal differential rank of Φ on every exact-rank stratum of Z; for ranks $0, \dots, 6$ the values are

$$0,\ 9,\ 18,\ 20,\ 20,\ 20,\ 20.$$

A hidden $O(2) \times O(2)$ action accounts for the generic source kernel at ranks one and two. Rank two is the maximal proper stratum: its image has dimension eighteen, whereas rank three is already locally surjective.

For $Q = Q_H + iQ_W$, written in 2×2 blocks $Q = \left(\begin{smallmatrix} P & C \\ C^{\mathsf{T}} & R \end{smallmatrix}\right)$, the rank-two image is locally cut out by two homogeneous degree-eight polynomials. On the chart $\det P \det R \neq 0$, they state that the trace and determinant of $P^{-1}CR^{-1}C^{\mathsf{T}}$ are real. For a normalized six-parameter pencil with determinant $q(s) = 650 + c_1 s + c_2 s^2 + c_3 s^3 + 5720 s^4$, these equations force

$$(c_1 - c_3)(80c_1 - 100c_2 + 50c_3 - 184573) = 0.$$

This coefficient obstruction is independent of any selected target quartic. As an exact corollary, two Bézout identities over $\mathbb{F}_7$ exclude all real spectral factors of a concrete quartic, even after complexification. Thus ranks at most two give exclusion, while ranks at least three restore local target freedom.

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1 Introduction

The positive symmetric elements of an indefinite orthogonal group form a classical noncompact Cartan cell. In graph coordinates they are parameterized by strict contractions, which makes low-rank restrictions on the graph map a natural source of nonlinear subvarieties. This paper studies a concrete projection of the balanced $O(6,6)$ cell to

$$\mathrm{Sym}_4(\mathbb{R}) \oplus \mathrm{Sym}_4(\mathbb{R}).$$

The projection comes from retaining two physical directions in a three-direction axis decomposition and four marked channel coefficients. The construction arose in a vector-spinor principal-symbol problem, but the main results are finite-dimensional matrix theorems.

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Fixed-rank matrix manifolds and their Stiefel/Grassmann coordinate descriptions are classical [6]. Quotient descriptions of bounded-rank matrix spaces continue to play a central role in matrix geometry [5]. General implicitization theory for rational maps is developed through symmetric and Rees algebras, approximation complexes, and Jacobian methods [3, 4]; contractive determinantal representations have also been studied on matrix Cartan domains [7], while determinantal rank strata on bounded symmetric domains appear naturally in the Jordan-theoretic setting [10]. The present map has a different form: a rank constraint is imposed on the graph coordinate of a real orthogonal Cartan cell, followed by the nonlinear Cartan transform and a fixed block projection. The claims below concern this concrete projected map; no general priority claim is made for arbitrary Cartan-domain implicitization.

The projection is not chosen by fitting the target equations. It is the real–imaginary compression obtained by retaining two directions from a three-axis decomposition and four marked channel coordinates: Q_H is the difference of the two diagonal axis blocks and Q_W is the symmetrized off-diagonal block. This is the smallest finite-dimensional model that preserves the split-signature Cartan constraint, the hidden transverse frame symmetry, and the matrix-pencil structure motivating the application.

The main results are organized in three steps. First we prove the complete rank hierarchy and identify the hidden compact kernel. Second we derive and locally complete the two equations of the rank-two image. Third we derive a coefficient-level obstruction for an entire normalized quartic family and apply it to a complete spectral-factor field. The geometric theorem is independent of that application.

First, how large is the image of each exact-rank contraction stratum? The source dimensions are standard, but the projection has a compact hidden symmetry and nontrivial rank behavior. We prove the complete hierarchy in modal coordinates. Ranks one and two both become locally finite over their images after quotienting the hidden frame rotations; rank two is the unique maximal proper stratum.

Second, can the rank-two image be implicitized? A singular-value factorization reveals a 2×2 modal matrix whose eigenvalues are positive real numbers. The reality of its trace and determinant gives two degree-eight target equations. Their gradients are independent at the rank-two witness, so they are the generic local equations predicted by the dimension count.

Third, what does the rank-two image force on normalized quartic completions? The family fixes two rows and leaves a symmetric 2×2 auxiliary block on each physical axis. For every determinant

$$q(s) = 650 + c_1 s + c_2 s^2 + c_3 s^3 + 5720 s^4,$$

the two implicit equations imply the coefficient-level obstruction

$$(c_1 - c_3)(80c_1 - 100c_2 + 50c_3 - 184573) = 0. \tag{1}$$

This statement belongs to the entire normalized fixed-endpoint family and is independent of any selected target quartic. We then apply it to the real spectral factors of

$$\mathcal{R}(z) = 32718400z^4 + 75657071z^3 + 68935965z^2 - 1218736z + 422500. \tag{2}$$

Two explicit Bézout identities over $\mathbb{F}_7$ prove that neither factor in (1) can vanish on the complete spectral-factor field. The rank-two image is therefore disjoint from the normalized spectral-factor family, even over $\mathbb{C}$.

The differential-rank hierarchy is proved below without finite-field Jacobians. The normalized completion elimination is reduced to the displayed coefficient formulas and can be checked by direct

expansion. The supplied scripts work over $\mathbb{Q}$, finite fields, and explicit quadratic extensions, print the certificate values appearing in the text, and fail on any mismatch. Independent standard-library audits reconstruct the modal submersion determinant, the compact completion identity, the fixed 2×2 radical blocks, and the modular Bézout certificates used below.

2 The balanced Cartan cell and the physical projection

Let $E = \mathbb{R}^3$ have ordered basis (e_H, e_W, e_N), and let $F = \mathbb{R}^4$ have ordered basis (f_1, f_2, f_3, f_4). Set

$$A_* = \operatorname{diag}(1, 1, -1, -1), \qquad S = I_E \otimes A_*.$$

Let $\mathcal{B}_0$ be the natural axis–channel order

$$(H,1),(H,2),(H,3),(H,4),(W,1),\dots,(N,4),$$

and let $\mathcal{B}_J$ be the grouped order

$$(H,1),(H,2),(W,1),(W,2),(N,1),(N,2);(H,3),(H,4),(W,3),(W,4),(N,3),(N,4).$$

In the basis $\mathcal{B}_J$, the form S is $J = \operatorname{diag}(I_6, -I_6)$. Let Π be the permutation matrix sending $\mathcal{B}_J$ to $\mathcal{B}_0$. For a strict contraction $Z \in M_6(\mathbb{R})$, meaning $\|Z\|_{\mathrm{op}} < 1$, define

$$\mathcal{G}(Z) = \begin{pmatrix} (I + ZZ^{\mathsf{T}})(I - ZZ^{\mathsf{T}})^{-1} & 2Z(I - Z^{\mathsf{T}}Z)^{-1} \\ 2(I - Z^{\mathsf{T}}Z)^{-1}Z^{\mathsf{T}} & (I + Z^{\mathsf{T}}Z)(I - Z^{\mathsf{T}}Z)^{-1} \end{pmatrix}. \tag{3}$$

Then $\mathcal{G}(Z)$ is symmetric positive definite and

$$\mathcal{G}(Z) J \mathcal{G}(Z) = J.$$

Conversely, every positive symmetric J-orthogonal matrix has a unique form (3); this is the graph chart for the positive Cartan cell of $O(6,6)$ [8].

Return to the natural axis–channel ordering by setting

$$\widetilde{\mathcal{G}}(Z) = \Pi \mathcal{G}(Z) \Pi^{\mathsf{T}},$$

and denote its 4×4 axis blocks by $G_{ab}(Z)$, $a, b \in \{H, W, N\}$. Define the physical projection

$$\Phi(Z) = (Q_H, Q_W), \qquad Q_H = G_{HH} - G_{WW}, \qquad Q_W = G_{HW} + G_{WH}. \tag{4}$$

Both matrices are real symmetric. The target has dimension twenty.

For $0 \le k \le 6$, let

$$\mathcal{D}_k^\circ = \{Z \in M_6(\mathbb{R}) : \operatorname{rank} Z = k,\ \|Z\|_{\mathrm{op}} < 1\}. \tag{5}$$

Lemma 2.1 (rank-stratum dimension)**.** *The manifold $\mathcal{D}_k^\circ$ has dimension $k(12-k)$. On the chart where the leading $k \times k$ minor is invertible, every point has a unique expression*

$$Z = \begin{pmatrix} I_k \\ X \end{pmatrix} M \begin{pmatrix} I_k & Y \end{pmatrix}, \qquad M \in GL_k(\mathbb{R}). \tag{6}$$

Proof. The chart contains $k^2 + 2k(6-k) = k(12-k)$ free parameters. These are the standard local coordinates on the rank-k determinantal manifold [2]. □

Lemma 2.2 (rank lifting)**.** *Let F be a C^1 map on an open subset of $M_6(\mathbb{R})$. If the restriction of dF_Z to $T_Z\mathcal{D}_k^\circ$ has rank at least r, then for every $\ell > k$ there are rank-ℓ points arbitrarily close to Z at which the restricted differential has rank at least r.*

Proof. Choose singular-vector coordinates in which $Z = \operatorname{diag}(s_1, \dots, s_k, 0, \dots, 0)$ and add nonzero entries $\epsilon_{k+1}, \dots, \epsilon_\ell$ in distinct null directions. For the resulting Z_ϵ, one has $\ker Z_\epsilon \subset \ker Z$ and $\operatorname{im} Z \subset \operatorname{im} Z_\epsilon$. The standard tangent-space formula

$$T_Z\mathcal{D}_k^\circ = \{E : E(\ker Z) \subset \operatorname{im} Z\}$$

therefore gives $T_Z\mathcal{D}_k^\circ \subset T_{Z_\epsilon}\mathcal{D}_\ell^\circ$, after identifying all tangent spaces with subspaces of $M_6(\mathbb{R})$. Choose r tangent directions whose image under dF_Z has a nonzero minor. The same directions remain tangent at Z_ϵ, and continuity preserves that minor for all sufficiently small ϵ. □

3 Hidden compact symmetry and the exact rank hierarchy

The physical projection omits the third axis. Let C_+ and C_- be the two positive and two negative channel coordinates belonging to e_N.

Proposition 3.1 (hidden compact invariance)**.** *The group*

$$\mathcal{K} = O(C_+) \times O(C_-) \cong O(2) \times O(2)$$

acts on every $\mathcal{D}_k^\circ$ by

$$Z \longmapsto (I_4 \oplus U_+)Z(I_4 \oplus U_-)^\mathsf{T},$$

and Φ is invariant. A generic orbit has dimension two.

Proof. Let $L = I_4 \oplus U_+$ and $R = I_4 \oplus U_-$. Formula (3) gives

$$\mathcal{G}(LZR^\mathsf{T}) = \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix} \mathcal{G}(Z) \begin{pmatrix} L^\mathsf{T} & 0 \\ 0 & R^\mathsf{T} \end{pmatrix}.$$

The blocks in (4) use only the first four positive and first four negative coordinates, so they are unchanged. On a dense open set the two infinitesimal rotations are independent. □

Lemma 3.2 (visible modal coordinates at ranks one and two)**.** *For $k \in \{1, 2\}$, quotienting the compact singular-vector data by the hidden $O(2) \times O(2)$ rotations gives local coordinates*

$$(X, Y, T) \in M_{2\times k}(\mathbb{C}) \times M_{2\times k}(\mathbb{C}) \times \operatorname{Diag}_k(\mathbb{R}),$$

with $T = \operatorname{diag}(t_1, \dots, t_k)$, $t_j > 1$. On an open set the physical map is

$$P = XX^\mathsf{T}, \qquad R = YY^\mathsf{T}, \qquad C = XTY^\mathsf{T}. \tag{7}$$

For $k = 1$ this map is immersive whenever $X, Y \neq 0$. For $k = 2$ it is immersive whenever X, Y are invertible and $t_1 \neq t_2$.

Proof. Write the compact singular-vector matrices as three real $2 \times k$ axis blocks. Their first two blocks combine into a complex $2 \times k$ visible matrix. For $k \leq 2$, after quotienting the hidden $O(2)$ action, this visible matrix ranges over the open set

$$\{A \in M_{4\times k}(\mathbb{R}) : I_k - A^\mathsf{T}A > 0\};$$

the hidden block is the positive square root of $I_k - A^{\mathsf{T}}A$. The same holds independently on the negative side. Multiplication of the jth visible columns by $\sqrt{2s_j^2/(1-s_j^2)}$ and the change $t_j = s_j^{-1}$ are local diffeomorphisms, giving the displayed coordinates and (7).

For $k = 1$, the equation $\dot{X}X^{\mathsf{T}} + X\dot{X}^{\mathsf{T}} = 0$ forces $\dot{X} = 0$ when $X \neq 0$; the same holds for Y, and then $\dot{C} = \dot{t}\, XY^{\mathsf{T}} = 0$ forces $\dot{t} = 0$.

For $k = 2$, suppose the differential of (7) vanishes. Set

$$A = X^{-1}\dot{X}, \qquad B = Y^{-1}\dot{Y}.$$

The equations $\dot{P} = \dot{R} = 0$ imply $A^{\mathsf{T}} = -A$ and $B^{\mathsf{T}} = -B$. Hence $A = aJ_2$ and $B = bJ_2$ for the standard complex skew matrix J_2. The mixed block equation becomes

$$AT + \dot{T} + TB^{\mathsf{T}} = 0.$$

Its diagonal part gives $\dot{T} = 0$, while its two off-diagonal entries give $at_2 - bt_1 = 0$ and $-at_1 + bt_2 = 0$. Since $t_1^2 \neq t_2^2$, one has $a = b = 0$. Thus the differential is injective. The calculation is complex linear and therefore also proves injectivity on the real source tangent. □

Lemma 3.3 (rank-three modal submersion)**.** *The restriction of Φ to $\mathcal{D}_3^\circ$ is a submersion at an explicit strict contraction. More precisely, put*

$$T_0 = \operatorname{diag}\left(2, \frac{5}{2}, 3\right),$$

$$X_0 = \frac{1}{4}\begin{pmatrix} i & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Y_0 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 2 \end{pmatrix}.$$

These data are realized by a rank-three contraction with singular values $(1/2, 2/5, 1/3)$, and the differential of the modal map

$$P = XX^{\mathsf{T}}, \qquad R = YY^{\mathsf{T}}, \qquad C = XTY^{\mathsf{T}} \tag{8}$$

has real rank twenty at (X_0, Y_0, T_0) on the rank-three source tangent.

Proof. For $t_j = s_j^{-1}$ set $a_j = 2/(t_j^2 - 1)$. At the displayed point,

$$(a_1, a_2, a_3) = \left(\frac{2}{3}, \frac{8}{21}, \frac{1}{4}\right).$$

If the jth columns of X_0 and Y_0 are divided by $\sqrt{a_j}$, the two visible singular-vector Gram deficiencies are both

$$I_3 - \operatorname{Re}(\alpha^*\alpha) = I_3 - \operatorname{Re}(\beta^*\beta) = \operatorname{diag}\left(\frac{29}{32}, \frac{107}{128}, 0\right).$$

They are therefore completed by two hidden real rows, so the data come from orthonormal singular frames. Since all $s_j < 1$, the resulting Z is a strict contraction.

Near this point, quotienting each hidden $O(2)$ frame identifies the visible rank-three data with the hypersurface on which the corresponding Gram deficiency has determinant zero. Its linearized condition is

$$\operatorname{Re}\dot{X}_{13} = -\frac{3}{16}\dot{t}_3, \qquad \operatorname{Re}\dot{Y}_{23} = -\frac{3}{16}\dot{t}_3.$$

The twenty variations used below have $\dot{t}_3 = \operatorname{Re}\dot{X}_{13} = \operatorname{Re}\dot{Y}_{23} = 0$, hence are tangent.

Differentiate (8) and order the twenty target coordinates by real and imaginary parts of the symmetric entries of P and R, followed by the four entries of C. On the source variables

$$\operatorname{Re}X_{11}, \operatorname{Im}X_{11}, \operatorname{Re}X_{12}, \operatorname{Im}X_{12}, \operatorname{Im}X_{13}, \operatorname{Re}X_{21}, \operatorname{Im}X_{21}, \operatorname{Re}X_{22}, \operatorname{Im}X_{22}, \operatorname{Re}X_{23},$$
$$\operatorname{Re}Y_{11}, \operatorname{Im}Y_{11}, \operatorname{Re}Y_{12}, \operatorname{Im}Y_{12}, \operatorname{Re}Y_{13}, \operatorname{Im}Y_{21}, \operatorname{Re}Y_{22}, \operatorname{Im}Y_{22}, t_1, t_2,$$

the selected Jacobian minor decomposes, after row and column permutations, into blocks of sizes $3, 10, 4, 3$. The two 3×3 determinants are -2^{-6}, the 4×4 determinant is $9\,2^{-10}$, and singleton expansion of the 10×10 block leaves the matrix

$$K = \begin{pmatrix} \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & \frac{5}{8} & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{5}{8} & 0 \end{pmatrix}, \qquad \det K = -\frac{11}{2^{10}}.$$

The removed singleton factors multiply to $-3/16$, so the 10×10 block has determinant $33\,2^{-14}$. Hence the full minor is

$$\left(-2^{-6}\right)\left(33\,2^{-14}\right)\left(9\,2^{-10}\right)\left(-2^{-6}\right) = -\frac{297}{2^{36}} \neq 0.$$

Thus the differential is onto the twenty-dimensional target. The displayed block factorization proves nonvanishing; Supplementary File S1 includes a direct rational-arithmetic audit. □

Theorem 3.4 (determinantal Cartan hierarchy)**.** *The maximal differential rank of* Φ *on* $\mathcal{D}_k^\circ$ *is*

k	0	1	2	3	4	5	6
$\max \operatorname{rank} d\Phi$	0	9	18	20	20	20	20.

(9)

The corresponding generic raw fiber dimensions are

k	0	1	2	3	4	5	6
$\dim \mathcal{D}_k^\circ$	0	11	20	27	32	35	36
$\dim \ker d\Phi$	0	2	2	7	12	15	16.

(10)

On a dense open subset of the rank-one and rank-two strata, the differential kernel is exactly tangent to the hidden $O(2) \times O(2)$ *orbit.*

Proof. By theorem 3.1, the generic differential rank is at most $k(12-k)-2$ for $k = 1, 2$. The target dimension gives the upper bound twenty for $k \geq 3$. The quotient dimensions at ranks one and two are nine and eighteen. By theorem 3.2, the induced quotient map is immersive on a nonempty open set, so these upper bounds are attained and the generic kernel is exactly the hidden two-dimensional orbit.

By theorem 3.3, the differential has full target rank at rank three. Applying theorem 2.2 proves full target rank on each stratum $k = 4, 5, 6$. Subtracting the ranks from the source dimensions gives (10). □

Corollary 3.5 (the maximal proper stratum)**.** *After quotienting the source-invisible compact frame rotations, ranks one and two are both locally dimension-matched with their target images. On the open modal-coordinate sets of theorem 3.2, the rank-one quotient map is locally nine-to-nine, while*

$$\mathcal{D}_2^\circ/(O(2)\times O(2)) \longrightarrow \Phi(\mathcal{D}_2^\circ)$$

is locally eighteen-to-eighteen. Rank two is the unique maximal proper stratum: its target image has codimension two, whereas rank three already has full target differential rank and at least five nongauge fiber dimensions after the same quotient.

4 The rank-two modal factorization

Set

$$Q = Q_H + iQ_W \in \mathrm{Sym}_4(\mathbb{C})$$

and use the channel splitting $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$:

$$Q = \begin{pmatrix} P & C \\ C^{\mathsf{T}} & R \end{pmatrix}, \qquad P = P^{\mathsf{T}}, \quad R = R^{\mathsf{T}}. \tag{11}$$

Define

$$p = \det P, \qquad r = \det R,$$

$$N_{\mathrm{tr}} = \mathrm{tr}\Big(\mathrm{adj}(P)C\,\mathrm{adj}(R)C^{\mathsf{T}}\Big), \qquad N_{\det} = (\det C)^2,$$

and the real homogeneous degree-eight polynomials

$$\mathcal{I}_{\mathrm{tr}}(Q) = \mathrm{Im}(N_{\mathrm{tr}}\overline{pr}), \qquad \mathcal{I}_{\det}(Q) = \mathrm{Im}(N_{\det}\overline{pr}). \tag{12}$$

On the chart $pr \neq 0$, these are equivalent to the reality of the trace and determinant of

$$\mathcal{K}(Q) = P^{-1}CR^{-1}C^{\mathsf{T}}. \tag{13}$$

Theorem 4.1 (implicit equations for the rank-two image)**.** *Every target in $\Phi(\mathcal{D}_2^\circ)$ satisfies*

$$\mathcal{I}_{\mathrm{tr}}(Q) = 0, \qquad \mathcal{I}_{\det}(Q) = 0.$$

At the rational target

$$P = R = I_2, \qquad C = \mathrm{diag}(5/3, 5/4), \tag{14}$$

the two gradients are independent. Consequently, near that target, the rank-two image is a relatively open subset of the smooth codimension-two complete intersection defined by these equations.

Proof. Take a compact singular-value decomposition

$$Z = \sum_{j=1}^{2} s_j u_j v_j^{\mathsf{T}}, \qquad 0 < s_j < 1.$$

Write $u_j = (u_{Hj}, u_{Wj}, u_{Nj})$ and $v_j = (v_{Hj}, v_{Wj}, v_{Nj})$, with every component in $\mathbb{R}^2$, and set

$$\alpha_j = u_{Hj} + iu_{Wj}, \qquad \beta_j = v_{Hj} + iv_{Wj}.$$

Put

$$a_j = \frac{2s_j^2}{1-s_j^2}, \qquad \tau_j = s_j^{-1},$$

$$X = \begin{pmatrix} \sqrt{a_1}\alpha_1 & \sqrt{a_2}\alpha_2 \end{pmatrix}, \quad Y = \begin{pmatrix} \sqrt{a_1}\beta_1 & \sqrt{a_2}\beta_2 \end{pmatrix}, \quad T = \operatorname{diag}(\tau_1, \tau_2).$$

The rank-one contribution of the jth singular mode to the visible positive-positive block is $a_j \alpha_j \alpha_j^{\mathsf{T}}$; its negative-negative contribution is $a_j \beta_j \beta_j^{\mathsf{T}}$; and its mixed contribution is $a_j \tau_j \alpha_j \beta_j^{\mathsf{T}}$. Summing the two modes gives

$$P = XX^{\mathsf{T}}, \qquad R = YY^{\mathsf{T}}, \qquad C = XTY^{\mathsf{T}}. \tag{15}$$

Here every transpose is bilinear, not Hermitian; this is essential because Q is complex symmetric. When X and Y are invertible,

$$\mathcal{K}(Q) = X^{-\mathsf{T}} T^2 X^{\mathsf{T}}.$$

Hence its trace and determinant are $\tau_1^2 + \tau_2^2$ and $\tau_1^2 \tau_2^2$, both real. Clearing denominators gives (12); polynomial continuity covers the remaining block charts.

For independence, use (14), so that $C = T = \operatorname{diag}(t_1, t_2)$ with $(t_1, t_2) = (5/3, 5/4)$. This target is realized by singular values $(3/5, 4/5)$: choose the visible singular vectors so that $X = Y = I_2$ and complete them by orthogonal hidden components. Under the imaginary diagonal variation $\dot{C} = i \operatorname{diag}(u, v)$ one obtains

$$d\mathcal{I}_{\mathrm{tr}} = 2(t_1 u + t_2 v), \qquad d\mathcal{I}_{\det} = 2(t_1 t_2^2 u + t_1^2 t_2 v).$$

The determinant of these two linear forms is

$$4t_1 t_2 (t_1^2 - t_2^2) \neq 0.$$

Hence their common zero set is smooth of dimension eighteen at this target. By theorem 3.2, the rank-two image differential also has rank eighteen there, so the image and the complete intersection agree locally.

□

Remark 4.2 (complexification). The functions in (12) are real polynomials in the twenty entries of (Q_H, Q_W). After clearing the Cartan denominators, their pullbacks vanish on a nonempty Euclidean-open subset of the real rank-two determinantal variety, hence vanish identically and continue to the complexified rational map. In later elimination statements, "complexification" means extension of these real coefficient polynomials; Im is not being interpreted as an analytic operation on complex auxiliary variables.

5 A normalized pencil family and a coefficient obstruction

Consider the symmetric matrices

$$H = \begin{pmatrix} -9/14 & \sqrt{10}/14 & \sqrt{105}/14 & 0 \\ \sqrt{10}/14 & -6/7 & \sqrt{42}/14 & 0 \\ \sqrt{105}/14 & \sqrt{42}/14 & a & b \\ 0 & 0 & b & c \end{pmatrix} \tag{16}$$

and

$$W = \begin{pmatrix} -33/35 & -\sqrt{10}/35 & 0 & 3\sqrt{14}/35 \\ -\sqrt{10}/35 & -6/7 & 0 & -3\sqrt{35}/35 \\ 0 & 0 & d & e \\ 3\sqrt{14}/35 & -3\sqrt{35}/35 & e & f \end{pmatrix}. \tag{17}$$

The first two rows are fixed; the auxiliary blocks are $H_A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ and $W_A = \left(\begin{smallmatrix} d & e \\ e & f \end{smallmatrix}\right)$. These fixed entries are specified independently of the Cartan rank condition and of the quartic introduced below. The application asks whether any completion of this preassigned affine pencil family lies in the rank-two image; no coefficient of the projection Φ is fitted to the quartic.

Proposition 5.1 (specialized implicit equations)**.** *For $Q = H + iW$ from* (16)–(17)*,*

$$\mathcal{I}_{\det}(Q) = \frac{27}{1000}\Delta_2, \qquad \mathcal{I}_{\mathrm{tr}}(Q) = -\frac{3}{200}\Delta_3,$$

where

$$\Delta_2 = 14ac - 3af - 14b^2 + 6be - 3cd - 14df + 14e^2 \tag{18}$$

and

$$\begin{aligned} \Delta_3 = {} & 24a^2c + 30a^2f - 24ab^2 - 60abe - 82ac^2 + 24ae^2 - 82af^2 \\ & + 82b^2c + 30b^2d - 41b^2f + 82bce - 48bde + 164bef \\ & - 41c^2d + 24cd^2 - 82ce^2 + 30d^2f - 30de^2 - 41df^2 + 41e^2f. \end{aligned} \tag{19}$$

Proof. The fixed blocks satisfy

$$p := \det P = -\frac{3}{10} + \frac{7}{5}i, \qquad \det C = -\frac{3\sqrt{3}}{10}i,$$

and a direct 2×2 multiplication gives

$$C^{\mathsf{T}} \operatorname{adj}(P) C = \begin{pmatrix} -3/4 - 3i/5 & 0 \\ 0 & 9/50 + 9i/25 \end{pmatrix}. \tag{20}$$

Write

$$R = \begin{pmatrix} a + id & b + ie \\ b + ie & c + if \end{pmatrix}.$$

Then $N_{\det} = (\det C)^2 = -27/100$, while cyclicity of trace and (20) give

$$N_{\mathrm{tr}} = (-3/4 - 3i/5)(c + if) + (9/50 + 9i/25)(a + id).$$

Substitution into (12), followed only by expansion of the scalar products with $\overline{p \det R}$, yields (18)–(19). Thus the specialized reduction is a literal 2×2 calculation rather than a generic symbolic-matrix step. □

For later use, we record a complete determinant parametrization. Set $X(s) = a + sd$, $Y(s) = b + se$, and $Z(s) = c + sf$.

Lemma 5.2 (fixed-row determinant identity)**.** *For the matrices above,*

$$10 \det(H + sW) = U(s)V(s) - (4s + 5)(2s + 1)Y(s)^2, \tag{21}$$

where

$$U(s) = (2s + 1)X(s) + \frac{3}{2}, \qquad V(s) = (4s + 5)Z(s) + \frac{9}{5}s^2.$$

Proof. Expand the determinant first in the auxiliary 2×2 block and complete the product. The identity is polynomial in the six auxiliary entries and may also be checked directly. □

Let $q(s)$ be a quartic. For parameters α, β, let U_α and V_β be the unique quadratics with leading coefficients α, β and interpolation data

$$U_\alpha(-1/2) = \frac{3}{2}, \qquad U_\alpha(-5/4) = \frac{32}{9} q(-5/4), \tag{22}$$

$$V_\beta(-5/4) = \frac{45}{16}, \qquad V_\beta(-1/2) = \frac{20}{3} q(-1/2). \tag{23}$$

The interpolation conditions make the numerator vanish at $s = -5/4$ and $s = -1/2$. Therefore

$$Q_{\alpha,\beta}(s) = \frac{U_\alpha(s)V_\beta(s) - 10q(s)}{(4s+5)(2s+1)} = q_2 s^2 + q_1 s + q_0 \tag{24}$$

is a quadratic polynomial.

Proposition 5.3 (complete fixed-row criterion)**.** *Over $\mathbb{C}$, the pair (16)–(17) can satisfy $\det(H + sW) = q(s)$ if and only if there are α, β for which $Q_{\alpha,\beta}$ is a square of a linear polynomial, equivalently $q_1^2 = 4q_2 q_0$. A reconstruction is*

$$d = \frac{\alpha}{2}, \quad a = U_\alpha(0) - \frac{3}{2}, \quad f = \frac{\beta - 9/5}{4}, \quad c = \frac{V_\beta(0)}{5},$$

with $b^2 = q_0$, $2be = q_1$, and $e^2 = q_2$.

Proof. Equation (21) shows that the remaining term must be $Y(s)^2$. Conversely, the displayed reconstruction makes $Q_{\alpha,\beta} = Y^2$ and recovers the determinant. □

Theorem 5.4 (coefficient-level rank-two obstruction)**.** *Write a general quartic with the fixed endpoint coefficients as*

$$q(s) = 650 + c_1 s + c_2 s^2 + c_3 s^3 + 5720 s^4,$$

and let α, β be the two leading coefficients in Eq. (22). Set

$$\theta = 6\alpha - 5\beta, \qquad B = 3280c_1 - 2000c_2 + 1000c_3 - 16576493,$$

$$S = 150c_1 - 700c_2 - 150c_3 + 4459189.$$

After the reconstruction in theorem 5.3, define

$$\ell_2 = 120\Delta_2, \qquad \ell_3 = -1440\Delta_3.$$

Then

$$\ell_2 = 63\theta + B, \tag{25}$$

$$\ell_3 = \frac{4}{35} S\ell_2 - \frac{16400}{7}(c_1 - c_3)(80c_1 - 100c_2 + 50c_3 - 184573). \tag{26}$$

In particular, if both rank-two equations vanish, then

$$(c_1 - c_3)(80c_1 - 100c_2 + 50c_3 - 184573) = 0. \tag{27}$$

Proof. Solve the two interpolation problems for U_α and V_β as in Eq. (22), divide $U_\alpha V_\beta - 10q$ by $(4s+5)(2s+1)$, and use $b^2 = q_0$, $2be = q_1$, $e^2 = q_2$. Substitution into Eqs. (18) and (19) gives the following coefficient table:

	coefficient of θ	constant term
ℓ_2	63	B
ℓ_3	$\frac{36}{5}S$	$\frac{4}{35}SB - \frac{16400}{7}(c_1-c_3)(80c_1-100c_2+50c_3-184573)$.

The first row is (25); eliminating θ between the two rows gives (26). This is an elementary polynomial identity in $c_1, c_2, c_3, \alpha, \beta$. □

6 A spectral-factor corollary

Let $\mathcal{R}$ be the quartic in (2), and set

$$K(x) = x^3 - 5403492x^2 - 243953250375312x + 1021159769572229461056. \tag{28}$$

Lemma 6.1 (complete normalized spectral factors)**.** *The cubic K has three simple real roots, isolated by*

$$x_0 \in (-15149282, -15149281), \quad x_S \in (4095962, 4095963), \quad x_L \in (16456810, 16456811).$$

For $x \in \{x_S, x_L\}$ and $\varepsilon \in \{\pm 1\}$, set

$$a_1 = \varepsilon\sqrt{x}, \qquad a_2 = \frac{x + 1218736}{1300}, \qquad a_3 = \frac{a_2^2 - 61499965}{2a_1},$$

and

$$q_{x,\varepsilon}(s) = 650 + a_1 s + a_2 s^2 + a_3 s^3 + 5720 s^4. \tag{29}$$

These four polynomials are exactly the real quartics with positive constant and leading coefficient satisfying $q(it)q(-it) = \mathcal{R}(t^2)$. Moreover, $q_{x,-\varepsilon}(s) = q_{x,\varepsilon}(-s)$, so there are two unoriented real spectral-factor orbits.

Proof. For $q(s) = \sum_{j=0}^{4} a_j s^j$, coefficient comparison gives $a_0 = 650$, $a_4 = 5720$ after an overall sign choice, and

$$a_1^2 - 2a_0 a_2 = -1218736,$$
$$a_2^2 + 2a_0 a_4 - 2a_1 a_3 = 68935965, \qquad a_3^2 - 2a_2 a_4 = 75657071.$$

Eliminating a_2, a_3 yields $(a_1^2 + 10278436)K(a_1^2) = 0$. The first factor is impossible over the reals. The discriminant of K is

$$5653076868832579270876068040857600000000000 > 0,$$

so K has three distinct real roots. Its values have opposite signs at the two endpoints of each displayed interval. Those three disjoint intervals therefore contain all three roots, and precisely two are positive. The first two equations give the displayed formulas, and the last reduces to

$$\frac{(x + 10278436)K(x)}{11424400000000x} = 0.$$

This proves completeness. Changing the sign of a_1 changes exactly the odd coefficients. □

Theorem 6.2 (rank-two spectral-factor exclusion)**.** *No real normalized completion* (16)–(17) *whose determinant is a spectral factor* (29) *belongs to the rank-two Cartan image. More strongly, the complexified normalized spectral-factor variety is disjoint from the complexified rank-two rational image wherever the Cartan map is defined.*

Proof. Apply theorem 5.4 with $(c_1, c_2, c_3) = (a_1, a_2, a_3)$. If both implicit equations vanish, then

$$(a_1 - a_3)(80a_1 - 100a_2 + 50a_3 - 184573) = 0. \tag{30}$$

On the spectral-factor field, $a_1^2 = x$ and

$$a_2 = \frac{x + 1218736}{1300}, \qquad a_3 = \frac{a_2^2 - 61499965}{2a_1}.$$

Since $a_1 \neq 0$, the first alternative in (30) is equivalent to

$$\overline{R}_1(x) = x^2 - 942528x - 102449623412304 = 0.$$

For the second alternative, multiplication by $2a_1$ gives

$$A(x) - a_1 B(x) = 0,$$

where

$$A(x) = x^2 + 7845472x - 102449623412304, \qquad B(x) = 5200x + 18814562000.$$

Eliminating the sign of a_1 gives

$$\begin{aligned} N_2(x) &= A(x)^2 - xB(x)^2 \\ &= x^4 - 11349056x^3 - 339019260721824x^2 \\ &\quad - 1961519047035394974976x + 1049592533732290789277858841 6. \end{aligned}$$

Reduce the three monic polynomials modulo 7:

$$K \equiv x^3 - 3x^2 - 2x + 1, \qquad \overline{R}_1 \equiv x^2 + x,$$

$$N_2 \equiv x^4 + 2x^3 + 3x^2 - 2x.$$

In $\mathbb{F}_7[x]$ one has

$$(2x + 1)K - 2x^2\overline{R}_1 = 1, \tag{31}$$

$$(3x^2 - x + 1)K + (2 - 3x)N_2 = 1. \tag{32}$$

Thus neither residual polynomial shares a root with K. Because all three integer polynomials are monic, a nonconstant common factor over $\mathbb{Q}$ would remain nonconstant after reduction modulo 7, contradicting (31)–(32). Both alternatives in (30) are therefore impossible on the complete spectral-factor field.

By theorem 4.2, the two necessary rank-two equations also hold on the complexified rational image, proving both claims. The exact interpolation and compact identity are independently reconstructed using only the Python standard library by `verify_compact_completion_syzygy.py`; `verify_normalized_reduction.py` separately reconstructs the fixed 2×2 radical reduction and the two modular Bézout identities. □

Corollary 6.3 (closed determinantal hierarchy)**.** *Ranks zero, one, and two are disjoint from every normalized spectral-factor completion above. Ranks three through six have full local target rank and positive-dimensional nongauge fibers. Thus no pure condition* $\operatorname{rank} Z \le k$ *selects this matrix-pencil family: below rank three there is exact exclusion, and at rank three and above there is local nonselection.*

Proof. The rank-zero and rank-one images lie in the closure of the rank-two image and satisfy the same polynomial equations. Apply theorem 6.2. The higher-rank claim is theorem 3.4. □

7 Discussion and scope

The rank-two stratum is mathematically distinguished as the highest-rank and highest-dimensional proper branch. Rank one is also locally dimension-matched after quotienting the source-invisible compact rotations, but its image is only nine-dimensional. The rank-two image is not merely lower-dimensional: on a dense chart it has a concrete modal interpretation and two explicit target equations. The exact exclusion in theorem 6.2 therefore rules out the strongest purely determinantal candidate, not a sampled or arbitrarily fitted subfamily.

The coefficient obstruction (27) is a separate output of the paper: it applies to every quartic in the normalized fixed-endpoint family, whether or not that quartic arises from spectral factorization. The large numerical quartic enters only in the final coprimality corollary.

The result is negative with respect to the motivating pencil, but positive as a classification theorem. It identifies a sharp boundary:

$$\operatorname{rank} Z \le 2 \quad \Longrightarrow \quad \text{proper algebraic image and exact exclusion,}$$

$$\operatorname{rank} Z \ge 3 \quad \Longrightarrow \quad \text{full local target freedom and nongauge fibers.}$$

Any source mechanism that selects the motivating quartic must therefore use more than the singular-value count of the Cartan contraction. Plausible next conditions would constrain the nonzero singular values or impose a polynomial identity on $Z^{\mathsf{T}}Z$ derived from an operator algebra. Such a condition must be justified independently of the desired target.

The paper does not claim a global description of the rank-two image outside the generic block chart, nor does it claim that the two degree-eight equations generate its global prime ideal. The complete-intersection statement is local at the certified smooth point. The normalized rows in (16)–(17) are fixed application data; the proofs in this article require no external derivation of them.

Verification scope. Every identity used in the proof is displayed in the article. The supplied programs independently reconstruct those calculations and serve as reproducibility and regression audits.

A Reproducibility protocol

Supplementary File S1 contains the exact verification programs and theorem certificates used for reproducibility. It is the only supplementary object needed to reproduce the computational checks in this article.

1. `verify_determinantal_cartan_strata.py` constructs rank-chart witnesses over $\mathbb{Q}$, checks strict contraction through exact Frobenius norms, differentiates the rational Cartan formula, and independently verifies the rank table and the hidden compact kernel. These Jacobians now serve only as regression checks; the complete hierarchy has modal proofs in the main text.

2. `verify_rank_two_implicit_spectral_factor_exclusion.py` reconstructs the two degree-eight invariants, their gradients, the specialized formulas (18)–(19), the complete determinant-completion family, and the spectral-factor exclusion.

The package-level driver `verify_supplement.py` runs the independent audits. In particular, `verify_compact_completion_syzygy.py` reconstructs the interpolation and Eqs. (25) and (26) using exact sparse Laurent arithmetic; `verify_core_independent.py` reconstructs the modal factorization and residual elimination by a separate dual-number route; and `verify_core_stdlib.py` uses only Python's standard library. The audit `verify_normalized_reduction.py` implements exact multiquadratic arithmetic, derives the compressed 2×2 formulas, and verifies Eqs. (31) and (32). Finally, `verify_rank_three_modal_submersion.py` checks the visible Gram deficiencies and the exact determinant $297/2^{36}$ using rational arithmetic.

The primary symbolic scripts require Python 3 and SymPy; the standard-library audits have no third-party dependency. They use no floating-point decision for any theorem statement. A nonzero modular minor proves that the corresponding rational determinant is nonzero because reduction modulo the chosen prime is a ring homomorphism.

Computational assistance and verification

OpenAI ChatGPT (GPT-5.6 Thinking) was used interactively during exploratory conjecture generation, symbolic derivation, proof decomposition, and the drafting and refactoring of verification code. The tool was not treated as a mathematical authority. The author independently reconstructed every theorem statement and displayed calculation as a human-readable argument, reviewed all generated material, and accepts responsibility for the analysis. The accompanying programs use exact arithmetic and include structurally independent implementations of the central computations.

Supplementary material

Supplementary File S1 is an exact verification package containing the Python programs, theorem certificates, dependency specification, and release verifier for the rank hierarchy, modal implicitization, normalized reduction, and modular coprimality calculations.

Data and code availability

The exact verification software and certificates underlying the displayed calculations are supplied as Supplementary File S1. No empirical or observational data were generated, and no additional research dataset exists.

CRediT authorship contribution statement

Aaron Nagy: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Writing–original draft, Writing–review and editing.

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During the preparation of this work, the author used OpenAI ChatGPT (GPT-5.6 Thinking) in order to support content organization and improve language and readability. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article. The separate computational-assistance statement above describes its exploratory research use and the authorial verification applied to that use.

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